ciency factors, transverse bending factors, and in-plane flange stress distributions. The solution presented reproduces previously accepted solutions for isotropic materials when the appropriate assumptions are made. This solution enables curved composite beams with thin flanges to be quickly analyzed.

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Finite Element Analysis of Long Cylindrical Shells

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Introduction

THE numerical analysis of long structural elements such as beams and cylindrical shells often requires large computational effort. In many cases, the analyst is only interested in a relatively small region in which most of the physical activity takes place. This region may be characterized by some "irregularities," such as geometrical and material nonlinearities, anisotropy, inhomogeneity and nonuniform loading. However, a standard approach necessitates the discretization of the entire beam or cylinder. Thus, part of the computation is superfluous as far as the analyst is concerned. An example is the stress analysis of a helicopter rotor blade, modeled as a long beam. Most of the blade behaves linearly, but near the root there may be a region of plasticity, large rotations and other sources of nonlinearities.

Moreover, in the bending analysis of a cylindrical shell the response typically has the nature of a boundary layer, outside of which the solution decays quickly. This means that only the segment of the cylinder including this boundary layer has to be taken into account in the numerical scheme. Unfortunately, the analyst has usually only a rough ideas as to where this layer ends, and a superfluous domain has to be included in the computation to insure a conservative analysis. Even if large finite elements are being used in the less interesting region, one must use a transition zone in which the elements are made gradually larger, in order to avoid significant local errors. All of these considerations lead to a discretization with many elements.

In this Note, we consider the problem of a long cylindrical shell loaded axisymmetrically. We show how to eliminate most of the domain in an exact manner, such that the remaining computational domain is small. Thus, only a small domain has to be discretized. Also, the ambiguity related to the length of the boundary layer is totally avoided. The procedure adopted here is similar to the one that has been used in Keller and Givoli[1] for the reduced wave equation and in Givoli[2] for the equations of elastostatics. In Givoli and Keller,[3] the general mathematical features of the method are explained in detail.

Finite Element Formulations

In applying the method to one-dimensional bending problems we consider three different finite element formulations. A discussion on these formulations can be found in Strang and Fix,[4] Carey and Oden,[5] and Hughes.[6]

As an example, we consider a cylindrical shell of length L clamped at both ends and loaded axisymmetrically. We assume that outside a small region near the end $x=L$ the cylinder is linear and uniform. We define the computational domain to be the small interval $[R,L]$, where $x=R$ is a point bounding the nonlinear or nonuniform portion of the cylinder. At the point $x=R$ we shall impose an exact boundary condition. This boundary condition is obtained by solving the problem analytically in the large interval $[0,R]$.

The first formulation that we consider is based on the following statement of the problem:

$$(Etc^2u'')+\frac{Et}{a^2}u=p \qquad R<x<L \tag{1}$$

$$u(L)=u'(L)=0 \tag{2}$$

and the exact boundary condition at $x=R$, which has the form

$$\begin{pmatrix} u'''(R) \\ u''(R) \end{pmatrix} = -\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}\begin{pmatrix} u(R) \\ u'(R) \end{pmatrix} \tag{3}$$

Here u is the lateral displacement, E the Young's modulus, p the distributed lateral load, t the cylinder thickness, a the cylinder radius, and c the reduced thickness defined by $c^2=t^2/12\,(1-\nu^2)$ where ν is Poisson's ratio. Since u''' and u'' are force quantities and u and u' are displacement quantities, the matrix $\boldsymbol{m}$ in Eq. (3) is the "edge stiffness matrix" in this case.

The resulting finite element formulation is of a C^1 type, in which the shape functions are required to possess a continuous first derivative. The Hermite cubic shape functions satisfy this requirement. The final linear system of finite element equations is $\boldsymbol{Kd}=\boldsymbol{F}$, where $\boldsymbol{d}$ is the vector of unknowns at the nodes, and the stiffness matrix $\boldsymbol{K}$ can be written as the sum

$$\boldsymbol{K}=\boldsymbol{K}^a+\boldsymbol{K}^b \tag{4}$$

Here $\boldsymbol{K}^b$ is the contribution to the stiffness matrix from the exact boundary conditions [Eq. (3)]. An explicit expression for K^b_{AB} is

$$\begin{pmatrix} K^b_{11} & K^b_{12} \\ K^b_{21} & K^b_{22} \end{pmatrix} = Etc^2|_{x=R}\begin{pmatrix} m_{11} & m_{12} \\ -m_{21} & -m_{22} \end{pmatrix} \tag{5}$$

All other entries in $\boldsymbol{K}^b$ are zero. We see that $m_{12}=-m_{21}$ is a necessary condition to maintain the symmetry of $\boldsymbol{K}$.

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The second finite-element formulation that we consider is based on the following strong form of the problem:

$$Etc^2 u'' = M \tag{6}$$

$$M'' + (Et/a^2)u = p \tag{7}$$

$$u(L) = u'(L) = 0 \tag{8}$$

and the exact boundary condition at $x = R$, which now has the form

$$\begin{pmatrix} u'(R) \\ M'(R) \end{pmatrix} = -\begin{pmatrix} \tilde{m}_{11} & \tilde{m}_{12} \\ \tilde{m}_{21} & \tilde{m}_{22} \end{pmatrix} \begin{pmatrix} u(R) \\ M(R) \end{pmatrix} \tag{9}$$

Here M is the bending moment, which is considered an independent unknown. This form of the problem results in a C^0 mixed finite element formulation. The expression for $\boldsymbol{K}^b$, the contribution from the exact boundary condition to the stiffness matrix, is

$$\begin{pmatrix} K_{11}^b & K_{12}^b \\ K_{21}^b & K_{22}^b \end{pmatrix} = -Etc^2|_{x=R} \begin{pmatrix} \tilde{m}_{21} & \tilde{m}_{22} \\ \tilde{m}_{11} & \tilde{m}_{12} \end{pmatrix} \tag{10}$$

All the rest of the components of $\boldsymbol{K}^b$ are zero. The symmetry $\tilde{m}_{11} = \tilde{m}_{22}$ guarantees the symmetry of $\boldsymbol{K}$.

The third finite element formulation starts from the theory that includes transverse shear deformation:

$$[\kappa Gt(\theta - d')]' + \frac{Et}{a^2}u = p \tag{11}$$

$$(Etc^2\theta')' - \kappa Gt(\theta - u') = 0 \tag{12}$$

$$u(L) = \theta(L) = 0 \tag{13}$$

and the exact boundary condition, which is now

$$\begin{pmatrix} u'(R) \\ \theta'(R) \end{pmatrix} = -\begin{pmatrix} \hat{m}_{11} & \hat{m}_{12} \\ \hat{m}_{21} & \hat{m}_{22} \end{pmatrix} \begin{pmatrix} u(R) \\ \theta(R) \end{pmatrix} \tag{14}$$

Here the two independent unknowns are the lateral displacement u and the rotation θ, G is the shear modulus, and κ is the shear correction factor. Note that since $u' = \theta - H/\kappa GA$ the first component in the left-hand side of Eq. (14) is neither a pure force nor a pure displacement quantity. The resulting finite element formulation is again of a C^0 type.

The explicit expression for $\boldsymbol{K}^b$ in this case is

$$\begin{pmatrix} K_{11}^b & K_{12}^b \\ K_{21}^b & K_{22}^b \end{pmatrix} = -\begin{pmatrix} \kappa Gt\hat{m}_{11} & \kappa Gt(1 + \hat{m}_{12}) \\ Etc^2\hat{m}_{21} & Etc^2\hat{m}_{22} \end{pmatrix} \tag{15}$$

and all of the other components are zero.

Exact Boundary Conditions

To find the matrices in Eqs. (3), (9), and (14), we have to solve appropriate problems in the domain $[0,R]$, analytically. We shall consider the case of a semi-infinite cylinder $(xe(-\infty R))$, although a finite cylinder with various boundary conditions at $x = 0$ can be treated as well.

The analytic solutions of the three problems yield the following matrices:

$$\boldsymbol{m} = \begin{bmatrix} 4\beta^3 & -2\beta^2 \\ 2\beta^2 & -2\beta \end{bmatrix} \tag{16}$$

$$\tilde{\boldsymbol{m}} = \begin{bmatrix} -\beta & -\dfrac{1}{2\beta Etc^2} \\ 2\beta^2 Etc^2 & -\beta \end{bmatrix} \tag{17}$$

$$\boldsymbol{m} = \begin{bmatrix} -\alpha_R + \alpha_I A/B & -\alpha_I/B \\ \alpha_I(B + A^2/B) & -\alpha_R - \alpha_I A/B \end{bmatrix} \tag{18}$$

The constant β appearing in Eqs. (16) and (17) is defined as

$$\beta = \frac{1}{\sqrt{2ac}} \tag{19}$$

In Eq. (18), which is based on the shear deformation theory,

$$\mu = \frac{\kappa Gt}{Etc^2} \tag{20}$$

$$\gamma = \cos^{-1}\frac{1}{2ac\mu} \qquad \left(0 < \gamma < \frac{\pi}{2}\right) \tag{21}$$

$$\alpha_R = \frac{1}{\sqrt{ac}}\cos\frac{\gamma}{2} \qquad \alpha_I = \frac{1}{\sqrt{ac}}\sin\frac{\gamma}{2} \tag{22}$$

$$A = \frac{\cos\gamma/2}{\sqrt{ac}}\left(\frac{1}{2ac\mu} + 2\sin^2\gamma/2\right) \tag{23}$$

$$B = \frac{\sin\gamma/2}{\sqrt{ac}}\left(\frac{1}{2ac\mu} + 2\cos^2\gamma/2\right) \tag{24}$$

It is easy to check that the stiffness matrices $\boldsymbol{K}^b$ are symmetric in all three cases.

Numerical Example

We present some numerical results for the cylinder problem with the shear deformation formulation. The model problem to be investigated is as follows. A semi-infinite cylinder is clamped at the end and subjected to a uniform external pressure p acting only on a segment of length L starting from that end. The domain is taken as $(-\infty, L)$ and, thus, the cylinder is loaded only along $(0,L)$. The parameters are $p = 4$, $Etc^2 = 1$, $\kappa Gt = 1$, $L = 1$; the radius of the cylinder is $a = 10$ and the reduced thickness is $c = 1$. The analytic solution of this

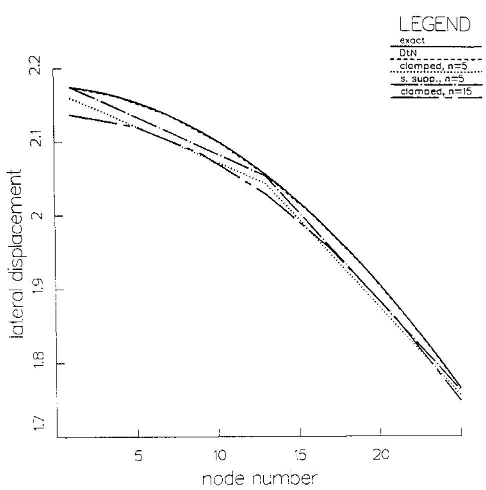

Fig. 1 The semi-infinite cylinder problem: comparison of various solutions.

problem can be found by solving a system of six linear algebraic equations.

In the current approach, we apply the exact boundary conditions [Eqs. (14) and (18)] at $R=0$. Thus the segment [0,1] is defined as the computational domain. A uniform mesh of 20 linear elements is used.

We would also like to solve the problem with the standard finite element procedure, and see how well we can do. To this end, we approximate the semi-infinite cylinder by a long finite cylinder extending from $x=-d$ to $x=L$. With the total number of elements in the mesh unchanged (20 elements), there are three parameters to be considered: the additional length d, the approximate boundary condtions at $x=-d$, and the distribution of the 20 elements in $(-d,L)$. We assume that there are n equal elements in $(0,L)$ and $20-n$ equal elements in $(-d,0)$. Several boundary conditions can be used at $x=-d$, namely $u=\theta=0$ (clamped), $H=\theta=0$ (guided), $u=M=0$ (simply supported), or $H=M=0$ (free).

Selected solutions for the normal displacement are compared in Fig. 1. These include the exact solution, the one using the exact boundary condition, and three solutions for the long finite cylinder with $d=10$: a clamped and a simply supported cylinder with $n=5$ and a clamped cylinder with $n=15$. The displacement is plotted in the interval [0,0.4]. The finite element solution using the exact boundary condition is very accurate, whereas the three other approximate solutions are much less so. Additional experiments show that inaccurate results are obtained by the standard finite element method for other values of d and n and for other boundary conditions at $x=-d$ as well.

Conclusion

A numerical scheme has been suggested for the solution of problems involving the bending of long axisymmetrically loaded cylindrical shells. By using an exact relation between the unknown variables and their derivatives, which in some cases is identical to the so-called "edge stiffness" relation, it is possible to treat most of the domain analytically. Thus, only a small computational domain is left for discretization. The exact relation is combined as a boundary condition with the finite element method. Numerical tests show that this procedure is both more accurate and more efficient than the standard one. A similar procedure can be applied to problems involving long beams. The application of the method to general elastic shells will be considered in future work.

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Buckling of Single Layer Composite Cylinders Subjected to Lateral Pressure

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Nomenclature

E, E_x, E_y	= Young's moduli
F_i	= parameters in stability equation, $i = 1 - 6$
$G_{xy} = G$	= shear modulus
k	= dimensionless shear modulus, $k = G(1-\nu_1\nu_2)/E_x$
L	= cylinder length
n	= number of circumferential buckle halfwaves
q, q_c	= lateral pressure, buckling lateral pressure
R	= cylinder radius
t	= cylinder thickness
U, V, W	= axial, circumferential, and radial displacements
x, y, z	= axial, circumferential, and radial coordinates
X	= material orthotropy parameter, $X^2 = \nu_1/\nu_2 = E_x/E_y$
α, λ, ϕ	= dimensionless parameters, $\alpha = t^2/12R^2$, $\lambda = \pi R/L$, $\phi = qR(1-\nu_1\nu_2)/E_x t$
$\epsilon_x, \epsilon_y, \gamma_{xy}$; $\sigma_x, \sigma_y, \sigma_{xy}$	= in-plane strains and stresses
ν, ν_1, ν_2	= Poisson's ratio
θ	= circumferential angle

I. Introduction

ONE of the main design criteria for cylinders in their engineering applications is the stability (buckling-resistance) when subjected to external loads of the following type: a) axial compression, b) shear forces, c) external uniform lateral pressure.

The analytic solutions for cases a) and c), for buckling of a cylinder made from an isotropic material, have been presented by Timoshenko and Gere,[1] Chaps. 11.1 and 11.5, respectively. Subsequently, for the same cases, Jones[2] developed analytic solutions for composite materials.

The purpose of the present effort has been to solve the buckling problem for single layer, composite, circular cylinders having simply supported edges, when subjected to an external uniform lateral pressure, case c). The composite is treated as an orthotropic material whose principal material directions are coincident with the axial and hoop directions.

Two buckling types, bending and shear, with different types of behavior were found. The analytic results were verified by numerical calculations.

II. Equilibrium Equations

We use in our presentation here the equilibrium differential equations derived in Ref. 1 Sec. 11.5(d) with respect to forces and moments acting in a cylindrical shell subjected to lateral pressure. Using the definitions of the forces and the moments per unit length

$$N_i = \int_{-t/2}^{t/2} \sigma_i \, dz, \qquad M_i = \int_{-t/2}^{t/2} z\sigma_i \, dz \tag{1}$$

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